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Distances in finite spaces from noncommutative geometry

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Abstract

Following the general principles of noncommutative geometry, it is possible to define a metric on the space of pure states of the noncommutative algebra generated by the coordinates. This metric generalizes the usual Riemannian one. We investigate some general properties of this metric in finite commutative cases corresponding to a metric on a finite set, and also compute explicitly some distances associated to commutative or noncommutative algebras. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

Though particle experiments are going further in energy and consequently deeper in the structure of matter, the geometric structure of space–time is still unknown. Classical differential geometry does not allow to take seriously into account both general relativity and quantum mechanics since the latest renounces intuitive geometric concepts while the first grounds its description of gravitation on purely geometric concepts. Different approaches of noncommutative differential geometry [1–3] give a mathematical framework for a geometric understanding of fundamental interactions. Saying geometric understanding, one would like

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to say clearer understanding: for instance, the noncommutative standard model [4–8] gives a geometric interpretation of the Higgs field together with an estimation on the mass of the corresponding boson.

But it is still difficult to draw an intuitive picture of a noncommutative space, as one can do for an Euclidean or even Riemannian space. A noncommutative space is described by a C^*-algebra $\mathcal{A}$, a faithful representation of $\mathcal{A}$ over a Hilbert space $\mathcal{H}$, and an operator D acting on $\mathcal{H}$. D has a compact resolvant and is possibly unbounded. To be precise, the algebra is restricted to the norm closure of the set of elements $a \in \mathcal{A}$ such that $[D, \pi(a)]$ is bounded.

A distance is then defined on $\mathcal{S}(\mathcal{A})$, state space of $\mathcal{A}$, by

$$d(\Phi, \Psi) = \sup_{a\in\mathcal{A}}\{|\Phi(a) - \Psi(a)|/\|[D, a]\| \leq 1\} \quad \forall\Phi, \Psi \in \mathcal{S}(\mathcal{A}). \tag{1}$$

If $\mathcal{A}$ is commutative, pure states correspond to characters. Thanks to Gelfand construction, they are interpreted as points, and $\mathcal{A}$ is the algebra of functions over these points. When $\mathcal{A}$ is not commutative, this interpretation is no more possible but the distance formula between pure states remains unchanged.

When $\mathcal{A}$ is the algebra of smooth functions over a Riemannian spin manifold, $\mathcal{H}$ the space of L^2-spinors and D the classical Dirac operator, then the noncommutative distance coincides with the geodesic distance. When $\mathcal{A}$ is tensorized by an internal algebra, for instance, the diagonal complex 2×2 matrices, then one obtains a space of two sheets with geodesic distance over each of them and a Kaluza–Klein geodesic distance between the two sheets [1]. Many works have been made on the dimension of algebras of functions over the noncommutative extension of space–time (see, for instance, Ref. [9] and its references, and Refs. [10,11]) and it would be interesting to compute the corresponding distances.

Even if this new geometry is a nice candidate for a better understanding of the gravity coupled with matter, including the standard model, it is plagued with its genetic Euclidean origin. Nevertheless, it is important to check if distances are calculable and if they can get a clear physical interpretation regarding with experiments comparison. We investigate here the first step of this program, paying particular attention to discrete spaces corresponding, for instance, to the internal degrees of freedom of fermions [5,6]. Previous noncommutative distances have been considered [12–14] in the case of finite algebras. Moreover, the classical distance in one-dimensional lattices can be obtained via the noncommutative approach [15,16].

In Refs. [17,18], the problem is introduced in a general framework: let L be a Lipschitz seminorm on a partially ordered real vector space A. L determines a metric ρ_L in the state space $\mathcal{S}(A)$:

$$\rho_L(\mu, \nu) = \sup_{a\in\mathcal{A}}\{|\mu(a) - \nu(a)|/L(a) \leq 1\} \quad \text{with} \;\; \mu, \nu \in \mathcal{S}(A).$$

ρ_L determines a Lipschitz seminorm L_{ρ_L} over the space $Af[\mathcal{S}(A)]$ of affine functions on $\mathcal{S}(A)$:

$$L_{\rho_L}(f) = \sup_{\mu,\nu\in\mathcal{S}(\mathcal{A})}\left\{\frac{|f(\mu) - f(\nu)|}{\rho(\mu, \nu)}/\mu \neq \nu\right\} \quad \text{with} \;\; f \in Af[\mathcal{S}(A)].$$

As A is isomorphic to a dense subspace of $Af[\mathcal{S}(A)]$, the question is: under which conditions one has $L_{\rho_L} = L$? Answers are given in Ref. [18]. In the noncommutative framework, taking $L(a) = \|[D, a]\|$, the question becomes: how to characterize the metrics ρ coming from a Dirac operator?

In this paper, we carry out the calculation of distances in spaces associated to finite-dimensional algebras. It seems natural to restrict to finite-dimensional representation of these algebras since, in the noncommutative approach of the standard model, the internal space is the space of fermions (more mathematical arguments to avoid infinite representations of finite-dimensional algebras can be found in Ref. [19]). Therefore, $\mathcal{A}$ is a direct sum of k matrix algebras, since any involutive algebra over $\mathbb{C}$ which admits a faithful finite-dimensional representation in a Hilbert space is a direct sum of matrix algebras. For $k = 1$, the simplest interesting case is $\mathcal{A} = M_2(\mathbb{C})$. The associated space is a fiber space whose base has only one point and the fiber is $\mathbb{C}^2$. For $k = 2$, we study the noncommutative space associated to $\mathcal{A} = M_p(\mathbb{C}) \oplus \mathbb{C},\ p \in \mathbb{N}$. This is a two-point space with fiber $\mathbb{C}^p$ over one of the point. Some applications can be found in Ref. [20], where $\mathcal{A} = M_2(\mathbb{C}) \oplus \mathbb{C}$ is used to build a first model of quantized space–time.

For $k \geq 3$, we restrict to commutative algebras. Then $\mathcal{A} = \mathbb{C}^k$ and $S(\mathcal{A})$ is simply a set of k points. We choose $\mathcal{H} = \mathbb{C}^k$. For the three-point space with any real self-adjoint operator D and the k point space with some particular operators D, we explicitly compute distances. To find a Dirac which gives a desired metric, it is enough to inverse formula. This is not possible in the four-point case for we show that generic distances are roots of polynomials which cannot be solved by radicals. In particular, this means that we have investigated all possible explicit distances for general k points because surprisingly counterexamples appear already for $k = 4$. A possible solution to overcome these difficulties could consist in modifying our definition of commutative spaces. For instance, using a slightly more complicated representation of $\mathcal{A}$ over a space $\mathcal{H}'$ larger than $\mathcal{H}$, one shows that there always exists an operator D' giving a set of given distances between the points. Moreover $(\mathcal{A}, \mathcal{H}', D')$ is a real spectral triple which fulfills all the axioms of noncommutative geometry.

2. Definition and notations

All along this paper, $\mathcal{A}$ is a unital C^*-algebra represented in a Hilbert space $\mathcal{H}$. D is a self-adjoint operator on $\mathcal{H}$ which does not belong to the commutant of $\mathcal{A}$. Its components are D_{ij}, $1 \leq i, j \leq n$. $\mathcal{A}_+$ is the subset of positive elements of $\mathcal{A}$ and $\mathcal{S}(\mathcal{A})$ its pure states space.

Lemma 1. *For any two states $\Phi, \Psi \in \mathcal{S}(\mathcal{A})$,*

$$d(\Phi, \Psi) = \sup_{a \in \mathcal{A}_+} \{|\Phi(a) - \Psi(a)| / \|[D, a]\| = 1\}.$$

Proof. Let $\theta \doteq \arg((\Phi - \Psi)(a_0))$, where $a_0 \in \mathcal{A}$ reaches the supremum in (1), namely

$$\|[D, a_0]\| \leq 1, \qquad |(\Phi - \Psi)(a_0)| = \mathrm{dist}(\Phi, \Psi).$$

The supremum is also reached for the self-adjoint element $b_0 = \frac{1}{2}(a_0\,e^{-i\theta} + a_0^*\,e^{i\theta}) \in \mathcal{A}$ since

$$\|[D, b_0]\| \leq \tfrac{1}{2}\|[D, a_0]\| + \tfrac{1}{2}\|[D, a_0^*]\| \leq 1,$$
$$|(\Phi - \Psi)(b_0)| = |\tfrac{1}{2}\mathrm{dist}(\Phi, \Psi) + \tfrac{1}{2}\overline{\mathrm{dist}(\Phi, \Psi)}| = \mathrm{dist}(\Phi, \Psi).$$

The same is true for $c_0 = b_0 + \|b_0\| I \in \mathcal{A}_+$, so we restrict to $\mathcal{A}_+$.

Suppose now $\|[D, c_0]\| < 1$. Take $e_0 \doteq c_0/\|[D, c_0]\| \in \mathcal{A}_+$, then

$$\|[D, e_0]\| = 1, \qquad |\Phi(e_0) - \Psi(e_0)| = \frac{|\Phi(c_0) - \Psi(c_0)|}{\|[D, c_0]\|} > |\Phi(c_0) - \Psi(c_0)|,$$

which is impossible since c_0 is chosen to reach the supremum. So $\|[D, c_0]\| = 1$.

If the supremum is not reached, the proof uses a sequence $\{a_n\}$ of elements of $\mathcal{A}$. □

- Once for all, any element $a \in \mathcal{A}$ that appears in a proof is self-adjoint.
- The canonical basis of $\mathbb{C}^n$ (or $\mathbb{R}^n$ in case) is denoted by $|1\rangle, |2\rangle, \ldots, |n\rangle$.
- When $\mathcal{A} = M_n(\mathbb{C})$, a pure state ω_ξ is determined by a normalized vector $\xi \in \mathbb{C}^n$: $\omega_\xi(a) = \xi^* a \xi \;\; \forall a \in M_n(\mathbb{C})$. Two normalized vectors determine the same pure state if and only if they are equal up to a phase. In other terms, $\mathcal{S}(M_n(\mathbb{C})) = \mathbb{C}P^{n-1}$.
- For any unitary operator U of $\mathcal{H}$ implementing $\mathcal{A}$, the jauge transformed of ω_ξ is $\tilde{\omega}_\xi(a) \doteq \omega_\xi(UaU^{-1}) \;\; \forall a \in \mathcal{A}$. If $\tilde{D} \doteq U^{-1}DU$, then $d_{\tilde{D}}(\tilde{\omega}_\xi, \tilde{\omega}_\zeta) = d_D(\omega_\xi, \omega_\zeta)$ for

$$\sup_{a \in \mathcal{A}_+} \{|(\tilde{\omega}_\xi - \tilde{\omega}_\zeta)(a)| / \|U^{-1}DU, a]\| = 1\}$$
$$= \sup_{UaU^{-1} \in \mathcal{A}_+} \{|(\omega_\xi - \omega_\zeta)(UaU^{-1})| / \|[D, UaU^{-1}]\| = 1\}.$$

3. One point space

The first nontrivial example with a single matrix algebra is $\mathcal{A} = M_2(\mathbb{C})$, represented in $\mathcal{H} = \mathbb{C}^2$ by

$$\mathcal{A} \ni a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

D is a 2×2 self-adjoint matrix with two real and distinct eigenvalues D_1, D_2. $\mathcal{S}(\mathcal{A}) = \mathbb{C}P^1$ is isomorphic to the sphere S^2. An explicit one-to-one correspondence is the Hopf fibration [21]: $\xi = (\xi_1, \xi_2) \in \mathbb{C}P^1$ is associated to $(a_\xi, b_\xi, c_\xi) \in S^2$ by

$$a_\xi \doteq 2\,\mathrm{Re}(\xi_1\bar{\xi}_2), \qquad b_\xi \doteq 2\,\mathrm{Im}(\xi_1\bar{\xi}_2) \qquad c_\xi \doteq |\xi_1|^2 - |\xi_2|^2.$$

Proposition 2. *$d(\omega_\xi, \omega_\zeta) = 2\sqrt{1 - |\langle\xi, \zeta\rangle|^2}/|D_1 - D_2|$ if $c_\xi = c_\zeta$, and is infinite otherwise.*

Proof. Let U be the unitary operator of $\mathcal{H}$ such that $\tilde{D}\doteq U^{-1}DU = \mathrm{diag}(D_1, D_2)$. A direct computation yields $\|[\tilde{D}, a]\| = |a_{12}||D_1 - D_2|$, thus the norm condition in (1) becomes

$$|a_{12}| \leq \frac{1}{|D_1 - D_2|}. \tag{2}$$

Furthermore, $(\omega_\xi - \omega_\zeta)(a) = \sum_{i,j=1}^{2} a_{ij}(\bar{\xi}_i\xi_j - \bar{\zeta}_i\zeta_j)$.

If $|\xi_1| \neq |\zeta_1|$, then a with all coefficients zero, except $a_{11} = L$, verifies the norm condition (2) for any $L \in \mathbb{R}^+$, and $|(\omega_\xi - \omega_\zeta)(a)| = L||\xi_1|^2 - |\zeta_1|^2|$. Thus, $d_{\tilde{D}}(\omega_\xi, \omega_\zeta) = +\infty$.

If $|\xi_1| = |\zeta_1|$, then $|\xi_2| = |\zeta_2|$ since $\|\xi\| = \|\zeta\| = 1$. So $c_\xi = c_\zeta$ and

$$|(\omega_\xi - \omega_\zeta)(a)| = |2\,\mathrm{Re}(a_{12}(\bar{\xi}_1\xi_2 - \bar{\zeta}_1\zeta_2))| \leq 2|a_{12}||(\bar{\xi}_1\xi_2 - \bar{\zeta}_1\zeta_2)|. \tag{3}$$

As any vector of $\mathbb{C}P^{n-1}$ is defined up to a phase, we assume that $\xi_1 = \zeta_1$ is real. Let $\theta_\xi \doteq \arg(\xi_2)$ and $\theta_\zeta \doteq \arg(\zeta_2)$. Take $a_{11} = a_{22} = 0$ and $\arg(a_{21}) = \frac{1}{2}(\pi - \theta_\xi - \theta_\zeta)$. Then

$$|\mathrm{Re}(a_{12}(\bar{\xi}_1\xi_2 - \bar{\zeta}_1\zeta_2))| = |a_{12}|\,|\xi_1|\,|\xi_2|\,|2\sin(\tfrac{1}{2}(\theta_\xi - \theta_\zeta))| = |a_{12}||\bar{\xi}_1\xi_2 - \bar{\zeta}_1\zeta_2|.$$

a reaches the upper bound in (3) and verifies the norm condition (2) as far as one chooses $|a_{12}| = 1/|D_1 - D_2|$. So $d_{\tilde{D}}(\omega_\xi, \omega_\zeta) = (2/|D_1 - D_2|)|\bar{\xi}_1\xi_2 - \bar{\zeta}_1\zeta_2|$. $\mathrm{Tr}(\xi\xi^* - \zeta\zeta^*)^2 = 2|\bar{\xi}_1\xi_2 - \bar{\zeta}_1\zeta_2|^2$. Developing the trace yields $1 - |\langle\xi, \zeta\rangle|^2 = |\bar{\xi}_i\xi_1 - \bar{\zeta}_i\zeta_1|^2$, thus $d_D(\omega_\xi, \omega_\zeta) = d_{\tilde{D}}(\tilde{\omega}_\xi, \tilde{\omega}_\zeta) = d_{\tilde{D}}(U^{-1}\xi, U^{-1}\zeta) = d_{\tilde{D}}(\omega_\xi, \omega_\zeta) = (2/|D_1 - D_2|)\sqrt{1 - |\langle\xi, \zeta\rangle|^2}$. □

We say that two states $\omega_\xi, \omega_\zeta \in \mathcal{S}(\mathcal{A})$ are at the same altitude if $c_\xi = c_\zeta$. By an easy calculation, for two such states, $d(\omega_\xi, \omega_\zeta) = (2/|D_1 - D_2|)\sqrt{(a_\xi - a_\zeta)^2 + (b_\xi - b_\zeta)^2}$. In other terms, up to a constant factor, d is nothing but the Euclidean distance restricted to planes of constant altitude. The distance between two planes of different altitude is infinite.

In a one point space with a fiber of higher dimension than $\mathbb{C}^2$, one needs an explicit formula for the norm of a self-adjoint $n \times n$ complex matrix. This is known to be generically impossible for $n \geq 5$.

4. Two-point space

Consider the algebra $\mathcal{A} = M_n(\mathbb{C}) \oplus \mathbb{C}$ represented on $\mathcal{H} = \mathbb{C}^n \oplus \mathbb{C}$ by

$$\mathcal{A} \ni a = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad x \in M_n(\mathbb{C}), \quad y \in \mathbb{C}.$$

The simplest interesting operator is

$$D = \begin{pmatrix} 0 & m \\ m^* & 0 \end{pmatrix}$$

with $m \in \mathbb{C}^n$ a nonzero column vector. $\mathcal{S}(\mathcal{A})$ is the union of the single pure state of $\mathbb{C}$, $\omega_0 \doteq \mathrm{Identity}$, with $\mathcal{S}(M_n(\mathbb{C}))$.

Proposition 3.

1. $$d(\omega_\xi,\omega_0)=\begin{cases}\dfrac{1}{\|m\|} & \textit{if } \xi \textit{ and } m \textit{ are colinear},\\ +\infty & \textit{otherwise}.\end{cases}$$

2. $$d(\omega_\xi,\omega_\zeta)=\begin{cases}\dfrac{2}{\|m\|}\sqrt{1-1|\langle\xi,\zeta\rangle|^2} & \textit{if } (\xi-\zeta\,\mathrm{e}^{\mathrm{i}\theta}) \textit{ and } m \textit{ are}\\ & \textit{colinear for some } \theta\in[0,2\pi[,\\ +\infty & \textit{otherwise}.\end{cases}$$

Proof. We may assume that $\|m\|=1$ for dividing D by $\|m\|$ means multiplying distances by $\|m\|$. Thus, there is a unitary operator $u\in M_n(\mathbb{C})$ such that $m=u|1\rangle$. With

$$U\doteq\begin{pmatrix}u & 0\\ 0 & 1\end{pmatrix},$$

$\tilde{D}\doteq U^{-1}DU$ and $z\doteq x-yI_n$ (self-adjoint by Lemma 1) one has

$$\|[\tilde{D},a]\|^2=\left\|\begin{pmatrix}0 & -z|1\rangle\\ \langle 1|z & 0\end{pmatrix}\right\|^2=\left\|\begin{pmatrix}z|1\rangle\langle 1|z^* & 0\\ 0 & \langle 1|zz^*|1\rangle\end{pmatrix}\right\|=\sum_{i=1}^{n}|z_{i1}|^2. \qquad (4)$$

Note that $\|[\tilde{D},a]\|^2=1$ implies $|z_{i1}|\le 1\ \ \forall i\in\{1,n\}$.

1. $(\omega_\xi-\omega_0)(a)=\xi^*x\xi-y=\xi^*z\xi=\sum_{i,j=1}^{n}\bar{\xi}_i z_{ij}\xi_j$.

   Assume $\xi_k\neq 0$ for $k\in\{2,n\}$. Take the matrix z with all coefficients zero except $z_{kk}=L\in\mathbb{R}^+$. By (4), z satisfies the norm condition of formula (1) and $|(\omega_\xi-\omega_0)(a)|=|\xi_k|^2L$. Thus, $d_{\tilde{D}}(\omega_\xi,\omega_{\xi_0})=+\infty$.

   Assume $\xi_i=0\ \forall i\in\{2,n\}$: there is constant θ such that $\xi=\mathrm{e}^{\mathrm{i}\theta}|1\rangle$. So $|(\omega_\xi-\omega_0)(a)|=|z_{11}|\le 1$. This upper bound is reached by z with all coefficients zero except $z_{11}=1$.

2. $(\omega_\xi-\omega_\zeta)(a)=\sum_{i,j=1}^{n}z_{ij}(\bar{\xi}_i\xi_j-\bar{\zeta}_i\zeta_j)$.

   Assume $(\bar{\xi}_p\xi_l-\bar{\zeta}_p\zeta_l)\neq 0$ for $p,l\in\{2,n\}$. The proof is similar as (1) with $z=0$ except $z_{pl}=L$.

   Assume $\bar{\xi}_i\xi_j-\bar{\zeta}_i\zeta_j=0\ \forall i,j\in\{2,n\}$. This is equivalent to $\xi_i=\zeta_i\,\mathrm{e}^{\mathrm{i}\theta}$ with θ a constant. In other words, $(\xi-\zeta\,\mathrm{e}^{\mathrm{i}\theta})\sim|1\rangle$. Furthermore, since $\|\xi\|^2=\|\zeta\|^2=1$, one has $|\xi_1|=|\zeta_1|$. Thus,

$$\left|\sum_{i,j=1}^{n}z_{ij}(\bar{\xi}_i\xi_j-\bar{\zeta}_i\zeta_j)\right|=2\left|\sum_{i=2}^{n}\mathrm{Re}(z_{i1}(\bar{\xi}_i\xi_1-\bar{\zeta}_i\zeta_1)\right|$$
$$\le 2\sqrt{\sum_{i=2}^{n}|z_{i1}|^2}\sqrt{\sum_{i=2}^{n}|\bar{\xi}_i\xi_1-\bar{\zeta}_i\zeta_1|^2}\le 2\sqrt{\sum_{i=2}^{n}|\bar{\xi}_i\xi_1-\bar{\zeta}_i\zeta_1|^2}.$$

   Take $z=0$ except $z_{1i}=|\bar{\xi}_i\xi_1-\bar{\zeta}_i\zeta_1|\left(\sum_{i=1}^{n}|\bar{\xi}_i\xi_1-\bar{\zeta}_i\zeta_1|^2\right)^{1/2}$. z reaches the upper bound and verifies the norm conditions. Thus, $d_{\tilde{D}}(\omega_\xi,\omega_\zeta)=2\sqrt{\sum_{i=2}^{n}|\bar{\xi}_i\xi_1-\bar{\zeta}_i\zeta_1|^2}$ and we conclude as in Proposition 2. □

The cases $\mathcal{A} = M_p(\mathbb{C}) \oplus M_q(\mathbb{C})$ is not studied here, neither is the space associated to a sum of three or more algebras with at least a noncommutative one. We focus on sums of commutative algebras. Then $\mathcal{A}$ is isomorphic to $\oplus_{i=1}^{k}\mathbb{C}$. The space associated to $k = 1$ has no interest. With $k = 2$, there is only one distance to compute which is equal to $1/|D_{12}|$. The generic cases $k = 3, 4$ with a real operator D, and some examples with $k = n \in \mathbb{N}$ are considered below. Before, we present general results on commutative finite spaces.

5. Commutative finite spaces

An n-point commutative space is determined by a triplet $(\mathcal{A}, \mathcal{H}, D)$ in which $\mathcal{A} = \oplus_1^n\mathbb{C}$ is represented over $\mathcal{H} = \mathbb{C}^n$ as a diagonal matrix:

$$\mathcal{A} \ni a = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ \vdots & a_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & a_n \end{pmatrix},$$

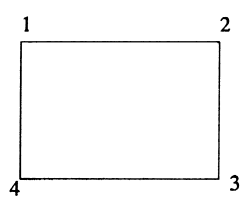
where $a_i \in \mathbb{C}$, but for distance computing we restrict to $a_i \in \mathbb{R}^+$ thanks to Lemma 1. To make computations easier, we only consider operators D with real entries. As D only appears through its commutator $[D, a]$, we assume that it has the following form:

$$D = \begin{pmatrix} 0 & D_{12} & \dots & \dots & D_{1n} \\ D_{12} & & D_{23} & & 0 \\ \vdots & D_{23} & 0 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & D_{n-1,n} \\ D_{1n} & \dots & \dots & D_{n-1,n} & 0 \end{pmatrix} \quad \text{with } D_{ij} \in \mathbb{R}.$$

Pure states can be interpreted as points of an n-point space whose function algebra is $\mathcal{A} : a(i) \doteq a_i$. The distance between two points i, j of this finite space is

$$d(i, j) = \sup_{a \in \mathcal{A}^+} \{|a_i - a_j| / \|[D, a]\| = 1\}. \tag{5}$$

In finite spaces, D may be interpreted as the adjacency matrix of a lattice [13]: two points i and j are connected if and only if $D_{ij} \neq 0$. For instance, in the four-point space, the restriction to $D_{13} = D_{24} = 0$ corresponds to a cyclic graph:



A path γ_{ij} is a sequence of p distinct points $(i, i_2, \dots, i_{p-1}, j)$ with $D_{i_k i_{k+1}} \neq 0\, \forall k \in \{1, p-1\}$. Since $d(1,2) = 1/|D_{12}|$ in a two-point space, the length of γ_{ij} is by definition

$$L(\gamma_{ij}) \doteq \sum_{k=1}^{p-1} \frac{1}{|D_{i_k i_{k+1}}|}.$$

Two points i, j are said connected if there exists at least one path γ_{ij}. The geodesic distance L_{ij} is by definition the length of the shortest path connecting i and j.

Proposition 4.

1. *Let D' be the operator obtained by canceling one or several lines and the corresponding columns of D. Then $d_{D'} \geq d_D$.*
2. *The distance between two points i and j depends only on the matrix elements corresponding to points situated on a path joining i and j.*
3. *The distance between two points is finite if and only if they are connected.*

Proof.

1. Let define $e \in \mathcal{A}$ by $e_i = 0$ if the ith line and column are cancelled, $e_i = 1$ elsewhere. e is a projection, commuting with $\mathcal{A}$, and $D' = eDe$. Thus, $\|[D', a]\| \leq \|[D, a]\|\, \forall a \in \mathcal{A}$. So

$$\sup\{|a_i - a_j| / \|[D, a]\| \leq 1\} \geq \sup\{|a_i - a_j| / \|[D', a]\| \leq 1\}.$$

2. Let Γ_{ij} denote the graph associated to the set of points belonging to any path γ_{ij}, and I_{ij} the set of points which are not on any path γ_{ij}. Any point of I_{ij} is connected by at most one path to Γ_{ij}. In other terms, $\forall l \in I_{ij}$ there is at most one point $m_l \in \Gamma_{ij}$ such that l and m_l are connected and $\gamma_{l m_l}$ has all its points (except m_l) in I_{ij}.

   Let D' be the operator obtained by cancellation of all lines and columns associated to points of I_{ij}. Assume that the supremum in (5) with D' is reached by $a' \in \mathcal{A} = \mathbb{C}^n$. Consider $a \in \mathbb{C}^n$ such that $a_p = a'_p$ except for the points of I_{ij} for which $a_l = a_{m_l}$ or $a_l = 0$ if m_l does not exist. Then, $\|[D, a]\| = \|[D', a']\|$, so $d_D(i, j) \geq d_{D'}(i, j)$. By (1), $d_D(i, j) \leq d'_D(i, j)$. Hence the result.
3. Suppose i and j are connected. There is at least one path $\gamma_{ij} = (i, i_2, \dots, i_{p-1}, j)$ whose length is the geodesic distance L_{ij}. Let obtain D' by canceling all lines and columns which do not correspond to points of γ_{ij}. Then $d(i, j) \leq d_{D'}(i, j)$. By the triangular inequality, one has

$$d_{D'}(i, j) \leq \sum_{k=1}^{p-1} d_{D'}(i_k, i_{k+1}) = \sum_{k=1}^{p-1} \frac{1}{|D_{i_k i_{k+1}}|} \doteq L_{ij},$$

   where $d(i, j)$ is smaller than the geodesic distance, thus it is finite.

If i and j are not connected, define $a \in \mathbb{C}^n$ by $a_i = t > 0$, $a_k = a_i$ if k and i are connected, $a_k = 0$ otherwise. Then $[D, a] = 0$ and $|a_i - a_j| = t$. As t is arbitrary, $d(i, j)$ is infinite. □

For simplification purpose, we write

$$a_{ij} \doteq a_j - a_i, \qquad x \doteq a_{21}, \qquad x_i \doteq a_{i+1,1}, \quad 2 \le i \le n-1. \tag{6}$$

In the $n = 3$ and $n = 4$ point case, this reduces to

$$y \doteq a_{31}, \qquad z \doteq a_{41}. \tag{7}$$

6. Distance on a regular space

An n-point commutative space is called regular when all coefficients of operator D are equal:

$$D = \{D_{ij}\} = \{k(1 - \delta_{ij})\}, \quad k \in \mathbb{R}.$$

Proposition 5.

1. *The distance between two points i, j of a regular space of constant k is*

$$d(i, j) = \frac{1}{|k|}\sqrt{\frac{2}{n}}.$$

2. *If the link, and only this link, between two-point i_1, i_2 is cut, $D_{i_1 i_2} = 0$, then*

$$d(i_1, i_2) = \frac{1}{|k|}\sqrt{\frac{2}{n-2}}.$$

Proof. In the regular space, the problem is symmetrical: all distances are equal and we compute $d(1, 2)$. When a link is cut, we take $i_1 = 1$, $i_2 = 2$ to fix notations, and denote by D' the operator D with $D_{12} = 0$. In both case, (5) and (6) yield

$$d(1, 2) = \sup_{a \in \mathcal{A}_+} \{|x| / \|[D \text{ or } D', a]\| = 1\}. \tag{8}$$

We first compute the norm of the commutator, and then find the supremum.

Lemma 6.

1. $$\|[D, a]\|^2 = |k|^2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} a_{ij}^2 = |k|^2 \left[x^2 + \sum_{i=2}^{n-1} \left(x_i^2 + (x - x_i)^2 + \sum_{j=i+1}^{n-1} (x_i - x_j)^2 \right) \right].$$

$$\|[D', a]\|^2$$

$$= \frac{|k|^2}{2}\left[\sum_{i=1}^{n}\sum_{\substack{j=i+1\\-(1,2)}}^{n} a_{ij}^2 + \sqrt{\left(\sum_{i=1}^{n}\sum_{\substack{j=i+1\\-(1,2)}}^{n} a_{ij}^2\right)^2 - 4a_{12}^2\sum_{i=3}^{n}\sum_{j=i+1}^{n} a_{ij}^2}\right]$$

$$= \frac{|k|^2}{2}\left[\sum_{i=2}^{n-1}\left(x_i^2 + (x - x_i)^2 + \sum_{j=i+1}^{n-1}(x_i - x_j)^2\right)\right]$$

$$+\left[\sqrt{\left(\sum_{i=2}^{n-1}\left(x_i^2+(x-x_i)^2+\sum_{j=i+1}^{n-1}(x_i-x_j)^2\right)\right)^2 - 4x^2\sum_{i=2}^{n-1}\sum_{j=i+1}^{n-1}(x_i-x_j)^2}\right].$$

2. *For the regular space, the supremum of x in* (8) *is reached when all x_i's are equal*.

Proof.

1. $C \doteq i[D, a]$ is the $n \times n$ matrix

$$C = k\begin{pmatrix} 0 & ia_{12} & & \\ ia_{21} & \ddots & ia_{ij} & \\ & ia_{ji} & \ddots & \\ & & & 0 \end{pmatrix}$$

with rank ≤ 2 since its kernel is generated by the $(n-2)$ independent vectors

$$\Lambda_k = \left(\frac{a_{k2}}{a_{21}}; \frac{a_{1k}}{a_{21}}; 0; \ldots; 1; \ldots; 0;\right), \quad 1 \text{ being at the } k\text{th position}, \; 3 \leq k \leq n.$$

Moreover C is Hermitian and traceless, so it has two nonzero real eigenvalues $\pm\lambda$. Thus, $\lambda = \sqrt{\frac{\mathrm{Tr}(C^2)}{2}}$. A direct computation yields $\lambda = k\sqrt{\sum_{i=1}^{n}\sum_{j=i+1}^{n}(a_{ij})^2}$. Finally, $\|[D, a]\| = \|i[D, a]\| = |\lambda| = |k|\sqrt{\sum_{i=1}^{n}\sum_{j=i+1}^{n}(a_{ij})^2}$ $= |k|\sqrt{x^2 + \sum_{i=2}^{n-1} x_i^2 + \sum_{i=2}^{n-1}\sum_{j=i+1}^{n-1}(x_i - x_j)^2}$.

Let $C' \doteq i[D', a]$. C' is the $n \times n$ matrix

$$C' = k\begin{pmatrix} 0 & 0 & ia_{13} & \\ 0 & 0 & & ia_{ij} \\ ia_{31} & & \ddots & \\ & ia_{ji} & & 0 \end{pmatrix}$$

with rank ≤ 4 since $\ker C'$ is generated by the $(n-4)$ independent vectors

$$\Lambda'_p = \left(0; 0; \frac{a_{p4}}{a_{43}}; \frac{a_{3p}}{a_{43}}; 0; \dots; 1; \dots; 0\right), \quad 1 \text{ being at the } p\text{th position}, \quad 5 \leq p \leq n.$$

Because C' is Hermitian and $\bar{C}' = -C'$, it has four real eigenvalues $\pm\lambda'_1, \pm\lambda'_2$. Thus, its characteristic polynomial is

$$\chi(C') = X^{n-4}(X^2 - {\lambda'_1}^2)(X^2 - {\lambda'_2}^2) = X^n - ({\lambda'_1}^2 + {\lambda'_2}^2)X^{n-2} + {\lambda'_1}^2{\lambda'_2}^2 X^{n-4}. \tag{9}$$

A direct computation yields

$${\lambda'_1}^2 + {\lambda'_2}^2 = \frac{1}{2}\mathrm{Tr}(C'^2) = k^2 \sum_{i=1}^{n} \sum_{\substack{j=i+1 \\ -(1,2)}}^{n} (a_{ij})^2.$$

The coefficient of X^{n-4} is the sum of all the minors of C' of degree 4. A minor $M(1,k,l,p)$ composed with the first (or second column) and three other columns $k, l, p \notin \{1,2\}$ (and the associated lines) is also a minor of C. As C is of rank ≤ 2, its minors of degree greater than 2 are null, so $M(1,k,l,p) = M(2,k,l,p) = 0$. The same is true for the minors $M(q,k,l,p)$ with $q \notin \{1,2\}$. Finally, the only nonzero minors are

$$M(1,2,l,p) = k^4 \,\mathrm{Det}\begin{pmatrix} 0 & 0 & ia_{1l} & ia_{1p} \\ 0 & 0 & ia_{2l} & ia_{2p} \\ ia_{l1} & ia_{l2} & 0 & ia_{lp} \\ ia_{p1} & ia_{p2} & a_{pl} & 0 \end{pmatrix} = k^4 \,\mathrm{Det}\begin{pmatrix} a_{1l} & a_{1p} \\ a_{2l} & a_{2p} \end{pmatrix}^2 = k^4 a_{21}^2 a_{pl}^2.$$

Summing all these minors gives ${\lambda'_1}^2{\lambda'_2}^2 = a_{12}^2 \sum_{l=3}^{n} \sum_{p=l+1}^{n} a_{pl}^2$. Then, solving (9) yields

$$\|[D', a]\|^2 = \frac{|k|^2}{2}\left(\sum_{i=1}^{n} \sum_{\substack{j=i+1 \\ -(1,2)}}^{n} a_{ij}^2 + \sqrt{\left(\sum_{i=1}^{n} \sum_{\substack{j=i+1 \\ -(1,2)}}^{n} a_{ij}^2\right)^2 - 4a_{12}^2 \sum_{i=3}^{n} \sum_{j=i+1}^{n} a_{ij}^2}\,\right).$$

2. Let $f(x, x_2, \ldots, x_{n-1}) \doteq x^2 + \sum_{i=2}^{n-1} x_i^2 + (x - x_i)^2 + \sum_{i=2}^{n-1} \sum_{j=i+1}^{n-1} (x_i - x_j)^2$ and suppose that $(x, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$ reaches the supremum, namely

$$f(x, x_2, \ldots, x_{n-1}) = \frac{1}{|k|^2}, \qquad d(1,2) = |x|,$$

then

2.1. x *is positive*: from the global parity of f, x can be chosen positive.

2.2. $x_i \leq \frac{1}{2}x\ \forall i \in \{2, \ldots, n-1\}$: suppose that p of the x_i's are greater than $\frac{1}{2}x$ and denote them generically by x_p. Consider now the $(n-1)$-uplet in which all x_p's are replaced by $\frac{1}{2}x$. Then, f decreases for $x_p^2 + (x - x_p)^2 \geq \frac{1}{4}x^2 + (x - \frac{1}{2}x)^2$ and $(x_i - x_p)^2 \geq (x_i - \frac{1}{2}x)^2$. Fixing the values of the remaining x_i's leads to see f as a function of the single variable x:

$$\begin{aligned} f(x) &= x^2 + p\left(\frac{x}{2}\right)^2 + \sum_i x_i^2 + p\left(x - \frac{x}{2}\right)^2 + \sum_i (x - x_i)^2 \\ &\quad + \sum_i p\left(\frac{x}{2} - x_i\right)^2 + \sum_i \sum_j (x_j - x_i)^2, \\ f'(x) &= 2x + 2px + 2\sum_i (x - x_i) + \sum_i p\left(\frac{x}{2} - x_i\right). \end{aligned}$$

As $x_i \leq \frac{1}{2}x \leq x$, $f'(x) > 0$ when $x > 0$. f is continue and $\lim_{x\to\infty} f(x) = +\infty$, so there is $x_0 > x$ such that $f(x_0) = 1/|k|^2$. In other terms, the initial $(n-1)$-uplet in x_p does not reach the supremum which is in contradiction with our hypothesis. So $p = 0$.

2.3. $x_i \geq 0\ \forall i \in \{2, \ldots, n-1\}$: the proof is the same by replacing all $x_i \leq 0$ by $\frac{1}{2}x$.

2.4. *All x_i's are equal*: let λ and Λ be the two smallest value of the x_i's with $\lambda \leq \Lambda$. If $\lambda = \Lambda$, then it comes immediately that all x_i's are equal. If $\lambda \neq \Lambda$, then

$$0 \leq \lambda < \Lambda \leq x_i \leq \tfrac{1}{2}x \quad \forall i \in \{2, \ldots, n-1\}.$$

Assume that m of the x_i's are equal to λ. Summing over $x_i \neq \lambda$, one obtains

$$\begin{aligned} f(x, x_2, \ldots, x_{n-1}) &= x^2 + m\lambda^2 + \sum_i x_i^2 + \sum_i (x - x_i)^2 + m(x - \lambda)^2 \\ &\quad + \sum_i m(\lambda - x_i)^2 + \sum_{i,j} (x_i - x_j)^2. \end{aligned}$$

Fix the values of $x_i \neq \lambda$ and consider now λ not like a constant but like the value of a variable $x_{\min}$. Then f can be seen as a function f_m of the two variables $x_{\min}$ and x with

$$\frac{\partial f_m}{\partial x_{\min}}(x_{\min}, x) = 2mx_{\min} + 2m(x_{\min} - x) + \sum_i 2m(x_{\min} - x_i).$$

As $(\partial f_m/\partial x_{\min})(x_{\min}, x) < 0$ for $x_{\min} \in [\lambda, \Lambda[$, one has $f_m(\Lambda, x) < f_m(\lambda, x) =$

$1/|k|^2$. Moreover,

$$\frac{\partial f_m}{\partial x}(\Lambda, x) = 2x + 2m(x - \Lambda) + \sum_i 2(x - x_i) > 0,$$

so there is $x_0 > x$ such that $f_m(x_0, \Lambda) = 1/|k|^2$, which is inconsistent with our hypothesis. So $\lambda = \Lambda$. □

Proof of Proposition 5.

1. According to Lemma 6, $x_i = x_2,\ 2 \le i \le n-1$. The norm condition in (8) becomes

$$2(n-2)x_2^2 + [2(2-n)x]x_2 + \left[(n-1)x^2 - \frac{1}{|k|^2}\right] = 0,$$

which has no real solutions in x_2 unless $|x| \le (1/|k|)\sqrt{2/n}$. This upper bound is reached when $x_2 = x/2 = (1/2|k|)\sqrt{2/n}$.

2. Let $h_1(x, x_i) \doteq \sum_{i=2}^{n-1} x_i^2 + (x - x_i)^2$, $h_0(x_i) \doteq \sum_{i=2}^{n-1} \sum_{j=i+1}^{n-1} (x_i - x_j)^2$, $g(x, x_i) \doteq h_1(x, x_i) - 2x^2$. Lemma 6 yields

$$\|[D', a]\|^2 = \tfrac{1}{2}|k|^2\left(h_1 + h_0 + \sqrt{h_1^2 + h_0^2 + 2gh_0}\right). \qquad (10)$$

Let $x_0 = \sup_{x, x_i \in \mathbb{R}}\{x / h_1(x, x_i) = 1/|k|^2\}$. As g and h_0 are both positive, (10) implies that $d(1, 2) \le x_0$. Imitating (1), one finds this upper bound is reached when all x_i's are equal, and $x_0 = (1/|k|)\sqrt{2/(n-2)}$. □

In finite spaces which are not regular, distances are not always explicitly computable. The cases $n = 3$ and $n = 4$ are considered below.

7. Three-point space

Proposition 7. *For a three-point space with operator*

$$D = \begin{pmatrix} 0 & D_{12} & D_{13} \\ D_{12} & 0 & D_{23} \\ D_{13} & D_{23} & 0 \end{pmatrix},$$

$$d(1, 2) = \sqrt{\frac{D_{13}^2 + D_{23}^2}{D_{12}^2 D_{13}^2 + D_{12}^2 D_{23}^2 + D_{23}^2 D_{13}^2}}, \qquad D_{ij} \in \mathbb{R},$$

the other distances come from suitable permutations of indices.

Proof. Eq. (5) and notations (7) gives

$$d(1, 2) = \sup_{a \in \mathcal{A}_+} \left\{ x / \|[D, a]\| = \left\| \begin{pmatrix} 0 & -D_{12}x & -D_{13}y \\ D_{12}x & 0 & D_{23}(x - y) \\ D_{13}y & D_{23}(y - x) & 0 \end{pmatrix} \right\| = 1 \right\}.$$

By direct calculation, $\|[D,a]\| = \sqrt{D_{23}(x-y)^2 + D_{13}y^2 + D_{12}x^2}$. Thus, $d(1,2)$ is the largest value of x for which there is a point (x,y) belonging to the ellipse

$$(D_{23}^2 + D_{12}^2)x^2 + (D_{13}^2 + D_{23}^2)y^2 - 2D_{23}^2 xy = 1. \tag{11}$$

$d(1,2)$ is the positive x for which the equation in y (11) has a zero discriminant, that is for

$$x = \sqrt{\frac{D_{13}^2 + D_{23}^2}{D_{12}^2 D_{13}^2 + D_{12}^2 D_{23}^2 + D_{23}^2 D_{13}^2}}. \qquad \square$$

The three distances verify an inequality of the triangle “power two” since

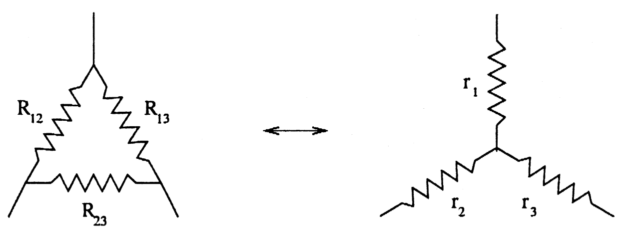
$$d(1,2)^2 + d(2,3)^2 \geq d(1,3)^2, \tag{12}$$

and two others inequality by permutations. The question rises of inverting the problem: for three positive numbers (a,b,c) verifying (12), is there any operator D giving (a,b,c) as noncommutative distances?

Proposition 8. *Let $a,b,c \in \mathbb{R}^+$ verifying $a^2+b^2 \geq c^2$, $b^2+c^2 \geq a^2$, $a^2+c^2 \geq b^2$. Then, there is an operator D such that $d(1,2) = a$, $d(1,3) = b$, $d(2,3) = c$, explicitly given by*

$$D_{12} = \sqrt{\frac{2(b^2+c^2-a^2)}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}},$$

where D_{13} *and* D_{23} *coming from permutations of* a,b,c.



Proof. Writing $1/D_{12}^2 = R_{12}$, $1/D_{23}^2 = R_{23}$, $1/D_{13}^2 = R_{13}$, Proposition 7 gives

$$\frac{1}{d(1,2)^2} = \frac{1}{R_{12}} + \frac{1}{R_{23}+R_{13}},$$

where $d(1,2)^2$ is the electrical resistance between points 1 and 2 of the triangle circuit. Finding D_{ij} means finding three elements R_{kp} that induce a resistance $d(i,j)^2$ between points i, and j. A classical result [22] indicates that the triangle circuit is equivalent to a stellar circuit (r_1,r_2,r_3) with

$$R_{12} = \frac{1}{r_3}(r_1r_2 + r_1r_3 + r_2r_3). \tag{13}$$

R_{13} and R_{23} coming from suitable permutations of indices. In the stellar circuit

$$\begin{aligned} d(1,2)^2 &= r_1 + r_2, & & 2r_1 = d(1,2)^2 + d(1,3)^2 - d(2,3)^2,\\ d(1,3)^2 &= r_1 + r_3, & \Rightarrow\quad & 2r_2 = d(1,2)^2 + d(2,3)^2 - d(1,3)^2,\\ d(2,3)^2 &= r_2 + r_3, & & 2r_3 = d(1,3)^2 + d(2,3)^2 - d(1,2)^2. \end{aligned}$$

Inserting in (13), this gives

$$D_{12} = \sqrt{\frac{2(d(1,3)^2 + d(2,3)^2 - d(1,2)^2)}{2(d(1,2)^2 d(1,3)^2 + d(1,2)^2 d(2,3)^2 + d(1,3)^2 d(2,3)^2) - d(1,2)^4 - d(1,3)^4 - d(2,3)^4}}. \qquad \square$$

8. Four-point space

Computing distances of an n-point space is endless. A priori the norm computation in Lemma 1 will be generically not possible for $n \geq 10$ since $[D, a]$ is an antisymmetric real matrix. However, even if for $n \leq 9$, the norm is always calculable, the convexity problem of Lemma 1 implies that the distance is not always calculable for $n \geq 4$. In this sense, this ends up the theory of explicit distances in n-point spaces.

Notations (7) are used as well as

$$d_1 \doteq \frac{1}{D_{12}}, \qquad d_2 \doteq \frac{1}{D_{13}}, \qquad d_3 \doteq \frac{1}{D_{14}}, \qquad d_4 \doteq \frac{1}{D_{23}}, \qquad d_5 \doteq \frac{1}{D_{24}}, \qquad d_6 \doteq \frac{1}{D_{34}}.$$

Theorem 9.

1. *$d(1,2)$ is a root of a polynomial of degree $\delta \leq 12$.*
2. *$d(1,2)$ is generically not solvable by radicals.*
3. *It is computable in the following case: when $1/d_2 = 1/d_5 = \infty$,*

$$d(1,2) = \begin{cases} d_1 & \text{if } d_1^2 \leq d_6^2, \ \text{else} \\[2mm] \dfrac{d_1\sqrt{(d_3^2 + d_1 d_6)^2}}{\sqrt{d_1^2 + d_3^2}\sqrt{d_3^2 + d_6^2}} & \text{if } d_1 d_6 = d_3 d_4, \ \text{else} \\[2mm] \sqrt{\dfrac{d_1^2(d_3^2 + d_6^2)(d_4^2 + d_6^2)}{(d_3 d_4 - d_1 d_6)^2}} & \text{if } C \leq 0, \ \text{else} \\[2mm] \max\left(\sqrt{\dfrac{d_1^2(d_3^2 + d_4^2)}{(d_3 + d_4)^2 + (d_1 - d_6)^2}}, \right. \\[2mm] \left. \sqrt{\dfrac{d_1^2(d_3^2 - d_4^2)}{(d_3 - d_4)^2 + (d_1 + d_6)^2}}\right), \end{cases}$$

where $C=((d_3+d_4)^2d_6+(d_1-d_6)(d_3d_4-d_6^2))((d_3-d_4)^2d_6+(d_1+d_6)(d_3d_4+d_6^2))$.

$$d(1,3)=\begin{cases}\sqrt{d_3^2+d_6^2} & \textit{if } (d_3^2+d_6^2)\le(d_1d_6-d_3d_4)^2,\\ \sqrt{d_1^2+d_4^2} & \textit{if } (d_1^2+d_4^2)\le(d_1d_6-d_3d_4)^2,\\ \max\left(\dfrac{\sqrt{(d_1d_3+d_4d_6)^2}}{\sqrt{(d_3+d_4)^2+(d_1-d_6)^2}},\right. & \\ \left.\dfrac{\sqrt{(d_1d_3+d_4d_6)^2}}{\sqrt{(d_3-d_4)^2+(d_1+d_6)^2}}\right) & \textit{otherwise}.\end{cases}$$

Permutations of the d_i*'s give* $d(2,3)$, $d(3,4)$, $d(1,4)$ (*resp.* $d(2,4)$) *from* $d(1,2)$ (*resp.* $d(1,3)$).

The rest of this section is devoted to the Proof of Theorem 9.

- $$[D,a]=\begin{pmatrix}0 & -\dfrac{x}{d_1} & -\dfrac{y}{d_2} & -\dfrac{z}{d_3}\\ \dfrac{x}{d_1} & 0 & \dfrac{x-y}{d_4} & \dfrac{x-z}{d_5}\\ \dfrac{y}{d_2} & \dfrac{y-x}{d_4} & 0 & \dfrac{y-z}{d_6}\\ \dfrac{z}{d_3} & \dfrac{z-x}{d_5} & \dfrac{z-y}{d_6} & 0\end{pmatrix},\qquad \vec{r}_a\doteq\begin{pmatrix}x\\ y\\ z\end{pmatrix}\quad \forall a\in\mathcal{A}_+=\mathbb{R}^{+4}.$$
- For $\vec{r}\doteq(x,y,z)\in\mathbb{R}^3$, define the functions:

$$\alpha(\vec{r})\doteq\frac{x^2}{d_1^2}+\frac{y^2}{d_2^2}+\frac{z^2}{d_3^2}+\frac{(x-y)^2}{d_4^2}+\frac{(x-z)^2}{d_5^2}+\frac{(y-z)^2}{d_6^2},$$
$$\beta(\vec{r})\doteq\frac{x(y-z)}{d_1d_6}+\frac{z(x-y)}{d_3d_4}+\frac{y(z-x)}{d_2d_5},$$
$$n(\vec{r})\doteq\alpha(\vec{r})+\sqrt{\alpha(\vec{r})^2-4\beta(\vec{r})^2},\ f(\vec{r})\doteq\alpha(\vec{r})-\beta(\vec{r})^2-1,$$

and the surfaces $\mathcal{N}$ and $\mathcal{F}$:

$$\mathcal{N}\doteq\{\vec{r}\in\mathbb{R}^3/n(\vec{r})=2\},\qquad \mathcal{F}\doteq\{\vec{r}\in\mathbb{R}^3/f(\vec{r})=0\}\quad\text{with }\ \mathcal{N}\subset\mathcal{F}.$$

Lemma 10.
1. *For* $a\in\mathcal{A}_+$, $\|[D,a]\|^2=\frac{1}{2}n(\vec{r}_a)$.
2. *For* $\vec{r}\in\mathcal{N}$ *such that* $\alpha(\vec{r})=2$, $\overrightarrow{\mathrm{grad}}(f)(\vec{r})=0$.

Proof.
1. The four eigenvalues of $i[D,a]$ are $\lambda_i=\pm(1/\sqrt{2})\sqrt{\alpha\pm\sqrt{\alpha^2-4\beta^2}}(\vec{r}_a)$, so

$$\|[D,a]\|^2=\tfrac{1}{2}\left(\alpha+\sqrt{\alpha^2-4\beta^2}\right)(\vec{r}_a).$$

2. We show that $(\partial f/\partial y)(\vec{r}) = 0$, the proof being the same for the other components of $\overrightarrow{\text{grad}}(f)$. As $\vec{r} \in \mathcal{N} \subset \mathcal{F}$ and $\alpha(\vec{r}) = 2$, $\beta(\vec{r}) = \pm 1$. If $\beta(\vec{r}) = 1$, then

$$\alpha(\vec{r}) = 2\beta(\vec{r}), \qquad \frac{\partial f}{\partial y}(\vec{r}) = \frac{\partial \alpha}{\partial y}(\vec{r}) - 2\beta(\vec{r})\frac{\partial \beta}{\partial y}(\vec{r}) = \frac{\partial \alpha}{\partial y}(\vec{r}) - 2\frac{\partial \beta}{\partial y}(\vec{r}).$$

Explicit calculation of $\alpha(\vec{r}) - 2\beta(\vec{r}) = 0$ shows that

$$\frac{x}{d_1} = \frac{y-z}{d_6}, \qquad \frac{y}{d_2} = \frac{z-x}{d_5}, \qquad \frac{z}{d_3} = \frac{x-y}{d_4},$$

which leads to $(\partial\alpha/\partial y)(\vec{r}) - 2(\partial\beta/\partial y)(\vec{r}) = 0$. The proof is the same if assuming $\beta(\vec{r}) = -1$. □

Thanks to notations and lemma above, (5) leads to

$$d(1,2) = \sup\{\langle 1|\vec{r}_a|1\rangle/\vec{r}_a \in \mathcal{N}\}. \tag{14}$$

This formula is not useful for $\mathcal{N}$ is not defined by a quadratic form. It is easier to work with $\mathcal{F}$.

Proposition 11. $d(1,2) \in \{\langle 1|\vec{r}|1\rangle/\vec{r} \in \mathcal{F}$ *and* $(\partial f/\partial y)(\vec{r}) = (\partial f/\partial z)(\vec{r}) = 0\}$.

Proof. The supremum in (14) is reached at a point $\vec{r}$ such that $\overrightarrow{\text{grad}}(n)(\vec{r})$, if it is defined, is parallel to the x axis. If $\alpha(\vec{r}) = 2$, then $\overrightarrow{\text{grad}}(n)(\vec{r})$ is not defined but $(\partial f/\partial y)(\vec{r}) = (\partial f/\partial z)(\vec{r}) = 0$ by Lemma 10. If $\alpha(\vec{r}) \neq 2$, then $\overrightarrow{\text{grad}}(f)(\vec{r})$ is collinear to $\overrightarrow{\text{grad}}(n)(\vec{r})$, so $(\partial f/\partial y)(\vec{r}) = (\partial f/\partial z)(\vec{r}) = 0$. To complete the proof, one just remarks that $\forall\vec{r} \in \mathbb{R}^3$, there is $a \in \mathcal{A}_+$ such that $\vec{r} = \vec{r}_a$, for instance, $a = (\xi, \xi - x, \xi - y, \xi - z)$ where $\xi \doteq \sup\{|x|, |y|, |z|\}$. □

According to this proposition, the distance is a common root of a polynomial in several variables and its various derivatives. Before undertaking explicit calculations, we recall general results about polynomial systems.

Notes on systems of polynomial equations. Let P and Q be two polynomials of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, \qquad Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$$

with $a_n, b_m \neq 0$. Without calculating the roots p_i, q_j of P, Q, one finds by algebraic manipulations [23] of the coefficients a_i and b_j the resultant of P and Q:

$$\text{Res}(P,Q) \doteq a_n^m b_m^n \prod_{i,j}(p_i - q_j), \quad 1 \leq i \leq n, \ \ 1 \leq j \leq m. \tag{15}$$

$\text{Res}(P,Q)$ is a polynomial in the a_i's and b_j's. P and Q have a common root if and only if their resultant is zero. A particular resultant is the discriminant:

$$\text{Dis}(P) \doteq \text{Res}(P, P').$$

P has a double root if and only if $\text{Dis}(P) = 0$. If P and Q are polynomials in x, y, z, then $\text{Res}[P, Q, y]$ denotes the resultant of P and Q seen as a polynomial in y. Equivalently, $\text{Dis}[P, z]$ stands for discriminant of P seen as a polynomial in z.

Proposition 12. *Let $P(x, y, z)$ be a polynomial of degree 2 in z, whose coefficients are real functions of x and y. If $P(x_0, y_0, z_0) = (\partial P/\partial y)(x_0, y_0, z_0) = (\partial P/\partial z)(x_0, y_0, z_0) = 0$ for some $(x_0, y_0, z_0) \in \mathbb{R}^3$, then x_0 is a root of the polynomial* $\mathrm{Dis}[\mathrm{Dis}(P, z), y]$, *and y_0 is a double root of the polynomial* $\mathrm{Dis}(P, z)(x_0, y)$.

Proof. Writing $P(x, y, z) = a(x, y)z^2 + b(x, y)z + c(x, y)$, a direct computation yields

$$V \doteq \mathrm{Dis}(P, z) = a(4ac - b^2), \qquad \frac{\partial V}{\partial y} = \frac{\partial a}{\partial y}(8ac - b^2) - 2ab\frac{\partial b}{\partial y} + 4a^2\frac{\partial c}{\partial y},$$

$$\mathrm{Res}\left(\frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, z\right) = b^2\frac{\partial a}{\partial y} - 2ab\frac{\partial b}{\partial y} + 4a^2\frac{\partial c}{\partial y}.$$

$P(x_0, y_0, z_0) = (\partial P/\partial z)(x_0, y_0, z_0)$ implies $V(x_0, y_0) = 0$, i.e. $(8ac - b^2)(x_0, y_0) = b^2(x_0, y_0)$, thus

$$\mathrm{Res}\left(\frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, z\right)(x_0, y_0) = \frac{\partial V}{\partial y}(x_0, y_0).$$

Therefore, $(\partial P/\partial y)(x_0, y_0, z_0) = (\partial P/\partial z)(x_0, y_0, z_0)$ implies $(\partial V/\partial y)(x_0, y_0) = 0 = V(x_0, y_0)$, thus y_0 is a double root of $V(x_0, y)$ and

$$\mathrm{Dis}(V, y)(x_0) = \mathrm{Dis}[\mathrm{Dis}[P, z], y](x_0) = 0. \qquad \square$$

Proof of Theorem 9. Propositions 11 and 12 yields

$$d(1, 2) \in \{x/\mathrm{Dis}[V(x, y), y] = 0\} \quad \text{with} \quad V(x, y) = \mathrm{Dis}[f(x, y, z), z].$$

Instead of $V(x, y)$, one uses the effective form without the corrective term $a_n^m b_m^n$ appearing in (15), so that zeros of V_{eff} correspond exactly to the existence of a common roots of f and $\partial f/\partial y$:

$$V_{\mathrm{eff}}(x, y) \doteq \mathrm{Numerator}\left(\frac{\mathrm{Dis}(f, z)}{n^n f_n^{2n-1}}\right)$$

with f_n the leading coefficient of f seen as a polynomial in z and $n = \deg f$. Note that the numerator is taken after a possible (but not always possible) simplification of the fraction.

1. By direct computation, $V_{\mathrm{eff}}(x, y) = V_i\, y^i$, $0 \le i \le 4$. Exact expressions of the V_i's are given in Appendix A. They are polynomial in x of the form: $V_4(x) = v_{4_0}$, $V_3(x) = v_{3_1}x$, $V_2(x) = v_{2_2}x^2 + v_{2_0}$, $V_1(x) = v_{1_3}x^3 + v_{1_1}x$, $V_0(x) = v_{0_4}x^4 + v_{0_2}x^2 + v_{0_0}$. The discriminant J of a polynomial $C = C_i\, y^i$ of degree four is

$$\begin{aligned} J(C) \doteq \mathrm{Res}[C, C'] = {} & C_4(C_3^2(C_1^2C_2^2 - 4C_1^3C_3 + 18C_0C_1C_2C_3 \\ & -C_0(4C_2^3 + 27C_0C_3^2)) + 2(-2C_2^3(C_1^2 - 4C_0C_2) \\ & +C_1C_2(9C_1^2 - 40C_0C_2)C_3 - 3C_0(C_1^2 - 24C_0C_2)C_3^2)C_4 \\ & -(27C_1^4 - 144C_0C_1^2C_2 + 128C_0^2C_2^2 \\ & +192C_0^2C_1C_3)C_4^2 + 256C_0^3C_4^3). \end{aligned}$$

   Replacing C_i by $V_i(x)$ shows that J is an even polynomial in x of degree $\delta \le 12$.

2. We compute an explicit counterexample: assume $d_1 = d_2 = d_3 = d_4 = d_6 = 1$ and $1/d_5 = 0$. Then

$$f(x, y, z) = x^2 + y^2 + z^2 + (x - y)^2 + (y - z)^2 - (x(y - z) + z(x - y))^2.$$

It is a polynomial of degree 2 in the variables x, y and z. Direct computation gives

$$V_{\text{eff}}(x, y) = 2 - 6y^2 + 3y^4 + 4x^2(-1 + y^2) - 4xy(-1 + y^2),$$
$$\text{Dis}(V_{\text{eff}}, y) = -768(-54 - 54x^2 + 135x^4 + 296x^6 - 368x^8 + 128x^{10}).$$

Let $p(x) = -128x^5 + 368x^4 - 296x^3 - 135x^2 + 54x + 54$. p has one real root x_1, two distinct complex ones x_2 and x_4 and their conjugates x_3 and x_5. Galois theory shows that p is not solvable by radical (for a comprehensive presentation see Ref. [24]):

p is irreductible over $\mathbb{Z}$ because it is irreductible over $\mathbb{Z}_5$. Indeed modulo 5 it becomes $q(x) = 2(x^5+4x^4+2x^3+2x+2)$ which has no roots in $\mathbb{Z}_5$. Therefore p is irreductible over $\mathbb{Q}$.

Let $E/\mathbb{Q}$ be a splitting field extension of p. As p has five distinct roots, its Galois group $\mathbb{G} = \text{Gal}(E/\mathbb{Q})$ is isomorphic to a subgroup of the symmetry group S_5 which is the permutation group of $X \doteq \{x_1, x_2, x_3, x_4, x_5\}$. As p has no repeated roots, p is separable so $|\mathbb{G}| = [E/\mathbb{Q}]$ where $|\mathbb{G}|$ denotes the order of $\mathbb{G}$ and $[E/\mathbb{Q}]$ is the index, i.e. the number of cossets $\mathbb{Q}$ in $\mathbb{G}$. If α is a root of p then $[\mathbb{Q}(\alpha), \mathbb{Q}] = 5$ so $|\mathbb{G}| = [E/\mathbb{Q}] = [E/\mathbb{Q}(\alpha)][\mathbb{Q}(\alpha), \mathbb{Q}]$ is divisible by 5. Thus, $\mathbb{G}$ contains an element of order 5: the 5-cycle $\tau = (12345)$.

The restriction to X of the complex conjugation gives rise to an element σ of $\mathbb{G}$: $\sigma = (23)(45)$. As σ is of order 2, $|\mathbb{G}|$ is divisible by 2. Moreover, $\tau\sigma = (124) \in \mathbb{G}$ is of order 3 which divides $|\mathbb{G}|$. Thus, $|\mathbb{G}|$ is a multiple of $5 \times 2 \times 3 = 30$ and divides $|S_5| = 120$. Since S_5 has no subgroup of order 30, $|\mathbb{G}| \in \{60, 120\}$. If $|\mathbb{G}| = 60$ then $\mathbb{G} = A_5$ but $\tau \notin A_5$. So $\mathbb{G} = S_5$.

S_n is solvable for $n \leq 4$ but is not solvable for $n \geq 5$, so $\mathbb{G}$ is not solvable. Then, by Galois theorem, p is not solvable by radicals.

3. When $d_2 = d_5 = 0$ and $d_1 d_6 \neq d_3 d_4$:

$$\begin{aligned}\text{Dis}(V_{\text{eff}}) = &-16 d_1^{16} d_3^{14} d_4^{12} d_6^{14} (d_4^2 + d_6^2)(x^2 - d_1^2)(x^2(d_3 d_4 - d_1 d_6)^2 \\ &- d_1^2 (d_3^2 + d_6^2)(d_4^2 + d_6^2))(x^2((d_3 - d_4)^2 + (d_1 + d_6)^2) \\ &- d_1^2 (d_3 - d_4)^2)^2 (x^2((d_3 + d_4)^2 + (d_1 - d_6)^2) - d_1^2 (d_3 + d_4)^2)^2.\end{aligned}$$

This polynomial has four single roots $\pm x_0, \pm x_1$ and four double roots $\pm x_2, \pm x_3$:

$$x_0 = |d_1|, \qquad x_1 = |d_1| \frac{\sqrt{(d_3^2 + d_6^2)(d_4^2 + d_6^2)}}{|d_3 d_4 - d_1 d_6|},$$

$$x_2 = |d_1| \sqrt{\frac{(d_3^2 + d_4^2)}{(d_3 + d_4)^2 + (d_1 - d_6)^2}}, \qquad x_3 = |d_1| \sqrt{\frac{(d_3^2 - d_4^2)}{(d_3 - d_4)^2 + (d_1 + d_6)^2}}.$$

By Propositions 11 and 12, $d(1, 2)$ is one of these x_i's, and the associated y_i is a double root of $V_{\text{eff}}(x_0, y)$. The corresponding z_i is determined by solving $f(x_i, y_i, z) = 0$.

Then one checks under which conditions each x_i verifies $n(x_i, y_i, z_i) = 2$ and finally take the greatest one. Considering x_0, one has $y_0 = d_1$, $z_0 = 0$, and

$$n(x_0, y_0, z_0) = 1 + \frac{d_1^2}{d_6^2} + \sqrt{\frac{(d_1^2 - d_6^2)^2}{d_6^4}} = \begin{cases} 2 & \text{if } d_1^2 \le d_6^2, \\ 2\dfrac{d_1^2}{d_6^2} > 2 & \text{if } d_1^2 > d_6^2. \end{cases}$$

Therefore, x_0 may be solution only if $d_1^2 \le d_6^2$. Likewise, with the corresponding y_1 and z_1 given in Appendix A, one checks that x_1 cannot be solution unless $C \le 0$. On the contrary, x_2 and x_3 may always be solutions for there are y_2, z_2 and y_3, z_3 such that $n(x_2, y_2, z_2) = n(x_3, y_3, z_3) = 2$ under no particular condition. By Proposition 2, canceling all links except d_1, $d(1,2) \le x_0$. So if $d_1^2 \le d_6^2$ then $d(1,2) = x_0$. As $x_1 \ge x_2$ and $x_1 \ge x_3$, $d(1,2) = x_1$ if $C \le 0$, otherwise $d(1,2) = \max(x_1, x_2)$. When $d_1 d_6 = d_3 d_4$, x_1 is not defined but the proof follows the same way.

Calculation of $d(1,3)$ is the same, except we are searching the maximum of y. $\mathrm{Dis}(V, x)$ is a polynomial in y of degree 12 with single roots $\pm y_0, \pm y_1$ and double roots $\pm y_2, \pm y_3$:

$$y_0 = \sqrt{d_3^2 + d_6^2}, \qquad y_1 = \sqrt{d_1^2 + d_4^2}, \qquad y_2 = d_1 \frac{|d_1 d_3 + d_4 d_6|}{\sqrt{(d_3 + d_4)^2 + (d_1 - d_6)^2}},$$

$$y_3 = \frac{|d_1 d_3 + d_4 d_6|}{\sqrt{(d_3 - d_4)^2 + (d_1 + d_6)^2}}.$$

With the associated x_i, z_i given in Appendix A, one checks that y_0 (resp. y_1) may be solution if $(d_3^2 + d_6^2)^2 \le (d_3 d_4 - d_1 d_6)^2$ (resp. $(d_1^2 + d_4^2)^2 \le (d_3 d_4 - d_1 d_6)^2$). As above, y_2 and y_3 may always be solution. Then, remark that $y_2, y_3 \le y_0$ and $y_2, y_3 \le y_1$. Finally, y_0 and y_1 cannot be simultaneous distinct solutions for adding both conditions yields $y_0 = y_1$. □

The four-point space shows that there is no hope to find a general formula for the metric in any commutative finite spaces: distances cannot be read directly in the Dirac operator through a finite algorithm. Computing the metric requires a more pragmatic approach and shall be undertaken case by case.

Consequently, the question of characterizing those metrics which come from a Dirac operator has still no answer: in a three-point space, one knows that the metrics satisfying (12) come from the Dirac operator given by Proposition 8, but this is no longer true in a four-point space for one does not have formulas to invert. However, a solution exists which consists in relaxing one of the constraint over our triplets, namely the choice of the representation space $\mathcal{H}$. Thus, as it is shown in the following section, for any given metric of an n-point space, one can build a corresponding Dirac operator. Moreover, the triplet obtained is then a spectral triple which satisfies the axioms of noncommutative geometry.

9. Distances and axioms of noncommutative geometry

In the previous discussion, we worked with triplets $(\mathcal{A}, \mathcal{H}, D)$ as if they satisfied all the axioms of noncommutative geometry. These axioms are introduced in order to recover the standard spin and Riemannian geometries in the commutative case [4]. Accordingly, for our distances to be bonafide noncommutative generalizations of Riemannian metrics, they have to be computed using triples satisfying all these axioms.

However, these axioms lead, in the finite case [25], to matrices whose size increases rapidly with n and thus prevents any computation except in few simple cases. This is the reason why we did not use these axioms up to now, but we shall see that the axioms do not put any constraints on the distances.

Proposition 13. *Let $(d_{ij})_{1\le i,j\le n,\, i\ne j}$ be any finite sequence of possibly infinite strictly positive numbers such that $d_{ij} = d_{ji}$ and $d_{ij} \le d_{ik} + d_{kj}$. Then there exists a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A} = \mathbb{C}^n$ satisfying all the axioms, and such that the resulting distance on the set of pure states of $\mathcal{A}$ is given by the numbers d_{ij}.*

To proceed, we shall first prove the following lemma.

Lemma 14. *There is a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A} = \mathbb{C}^n$ satisfying all the axioms such that*

$$\|[D, \pi(a)]\| = \sup_{1\le i,j\le n,\, i\ne j} \frac{|a_i - a_j|}{d_{ij}}, \tag{16}$$

where $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ and π denotes the representation of $\mathcal{A}$ on $\mathcal{H}$.

Proof. The proof is by induction on n.

The first nontrivial case is $n = 2$. We take $\mathcal{A}_2 = \mathbb{C}^2$ and $\mathcal{H}_2 = \mathbb{C}^3$. The representation π_2 and the chirality χ_2 are both diagonal and are given by $\pi_2(x_1, x_2) = \mathrm{diag}(x_1, x_2, x_2)$ and $\chi_2 = \mathrm{diag}(1, -1, 1)$. The Dirac operator D_2 and the charge conjugation $\mathcal{J}_2$ are defined as

$$\mathcal{D}_2 = \begin{pmatrix} 0 & \frac{1}{d_{12}} & 0 \\ \frac{1}{d_{12}} & 0 & \frac{1}{d_{12}} \\ 0 & \frac{1}{d_{12}} & 0 \end{pmatrix}, \qquad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} C,$$

where C is the complex conjugation and we set $1/d_{12} = 0$ if $d_{12} = \infty$.

In the finite case, all axioms reduce to the *reality*, *first-order*, *orientability* and *Poincaré duality* axioms [25]. In the present case, the first two are commutation relations easy to check due to the commutative nature of the algebra. The orientability axiom is fulfilled by writing the chirality as

$$\chi_2 = \pi_2(1, -1)\mathcal{J}_2\pi_2(-1, 1)\mathcal{J}_2^{-1}.$$

The multiplicity matrix is

$$\mu_2 = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$

which is nondegenerate and thus Poincaré duality holds. Finally, one easily checks that

$$\|[\mathcal{D}_2, \pi_2(x)]\| = \frac{|x_1 - x_2|}{d_{12}}.$$

Let us now assume that $(\mathcal{A}_n, \mathcal{H}_n, D_n)$ together with π_n, χ_n and $\mathcal{J}_n$ have been constructed for $n > 2$. To build the order $n+1$ spectral triple, we merely imitate the $n = 2$ construction. Let us take $\mathcal{A} = \mathbb{C}^{n+1}$ and

$$\mathcal{H}_n = \mathcal{H}_{n-1} \oplus \left(\bigoplus_{i=1}^{n-1} \mathcal{H}_n^i \right)$$

with $\mathcal{H}_n^i = \mathbb{C}^3 \, \forall i, n$. With respect to the previous decomposition all operators $\mathcal{O}$ corresponding to $\mathcal{D}, \pi, \chi$ and $\mathcal{J}$ are block diagonal and defined inductively as

$$\mathcal{O}_n = \mathcal{O}_{n-1} \oplus \left(\bigoplus_{i=1}^{n-1} \mathcal{O}_n^i \right).$$

As in the $n = 2$ case, we define

$$\pi_n^i(x_i, x_n) = \mathrm{diag}(x_i, x_n, x_n), \quad \chi_n^i = \mathrm{diag}(1, -1, 1).$$

The Dirac operator D_n and the charge conjugation $\mathcal{J}_n$ are defined as

$$D_n^i = \begin{pmatrix} 0 & \frac{1}{d_{in}} & 0 \\ \frac{1}{d_{in}} & 0 & \frac{1}{d_{in}} \\ 0 & \frac{1}{d_{in}} & 0 \end{pmatrix}, \qquad \mathcal{J}_n^i = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} C.$$

Then it is easy to check that all axioms but Poincaré duality hold using the induction assumption and the block diagonal nature of the construction.

The multiplicity matrix of this spectral triple is

$$\mu_n = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & -(n-1) \end{pmatrix}$$

If N is any positive integer, we can always consider the trivial spectral triple $(\mathbb{C}^n, \mathbb{C}^n \oplus \mathbb{C}^N, 0)$ with obvious representation and charge conjugation and whose chirality is equal to -1. If we take the direct sum of this spectral triple with $(\mathcal{A}_n, \mathcal{H}_n, D_n)$, the resulting multiplicity matrix is $\mu_n + NI_n$, which is nondegenerate for N sufficiently large. Accordingly, Poincaré duality will be satisfied.

Finally, the computation of the norm of the commutator $\|[\mathcal{D}_n, \pi_n(a)]\|$ follows easily from the block diagonal structure and the induction assumption. □

To complete the Proof of Proposition 13, we use the previous lemma to construct a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ fulfilling condition (16). If a verifies the norm condition, then $|a_i - a_j| \leq d_{ij}$, so $d(i, j) \leq d_{ij}$.

Furthermore, if we fix any two points such that $d_{ij} < \infty$ and take $x_k = d_{ik}$ (which is finite thanks to triangular inequality) one has $|x_i - x_j| = d_{ij}$. Another application of the triangular inequality yields $|d_{ik} - d_{il}| \leq d_{kl}$ for any k and l so that $\|[\mathcal{D}, \pi(x)]\| \leq 1$ by (16). This shows that $d_{ij} \leq d(i, j)$ so that the equality holds when d_{ij} is finite.

If d_{ij} is infinite, so are d_{ik} and d_{jk} for any k. Thus, the inequality (16) does not constraint x_i and x_j since the corresponding matrix element of D vanish. Therefore, we can send $|x_i - x_j|$ to infinity and we also have $d(i, j) = d_{ij}$.

10. Conclusion

As a conclusion of previous discussion, we may say that once given $\mathcal{A} = \mathbb{C}^n$, there is no constraint arising from the axioms of noncommutative geometry. Such constraints may only appear if one imposes some extra conditions, such as fixing $\mathcal{H} = \mathbb{C}^n$ as we did in the discussion of the three- and four-point cases. We stress that we only showed that the map which associates a metric to a Dirac operator is surjective. In a discrete analogue of a quantum theory of gravity based on eigenvalues of the Dirac operators [26], one also needs to know how many Dirac operators correspond to a given metric, as well as the possible relations between their spectra.

A naive question remains unanswered: what does these distances mean? According to spectral action principle [27], Dirac operator encodes both physics and metrics. In the standard model, computation of the action leads to the Lagrangian of the full standard model and the Einstein–Hilbert action of general relativity. So the coding of physics makes sense. But what about the coding of metrics? For the time being, the answer is clear in the simple continuous case where noncommutative distance is the geodesic distance. Distance in the geometry of standard model should have physical meaning, but it has not been explicitly computed yet. Fortunately, the complications due to Theorem 9 should not appear since the noncommutative standard model involves commutative algebras tensorized by $M_n(\mathbb{C})$ with n less than 3.

Appendix A. Coefficients of $V_{\text{eff}}(x, y)$ in the general four-point case

$$V_4(x) = \frac{4(d_3d_4 - d_2d_5)^2(d_4^2d_6^2 + d_2^2(d_4^2 + d_6^2))}{d_2^4d_3^2d_4^4d_5^2d_6^2},$$

$$V_3(x) = \frac{8x(d_2d_5 - d_3d_4)(d_3d_4d_5d_6(d_2^2 + d_4^2) + d_1(d_2d_3d_4(d_4^2 + d_6^2) - d_4^2d_5d_6^2 - d_2^2d_5(d_4^2 + 2d_6^2)))}{d_1d_2^3d_3^2d_4^4d_5^2d_6^2},$$

$$V_2(x) = \frac{4x^2}{d_1^2 d_2^2 d_3^2 d_4^4 d_5^2 d_6^2}[d_4^2(d_3^2 d_4^2 d_5^2 + d_1^2(d_3 d_4 - d_2 d_5)^2 + d_2^2(d_3^2 d_4^2 + (d_3^2 + d_4^2) d_6^2))$$
$$-2d_1 d_4 d_6 (d_2 d_4 d_5 (d_4^2 - 2d_3^2) + d_3 d_4^2 d_5^2 + d_2^2 d_3 (d_4^2 + 3d_5^2))$$
$$+d_6^2(d_4^2(d_3 d_4 - d_2 d_5)^2 + d_1^2(d_4^2(d_3^2 + d_4^2 + d_6^2)$$
$$-6d_2 d_3 d_4 d_5 + d_2^2(d_4^2 + 6d_6^2)))]$$
$$- \frac{4(d_2^2(d_3^2(d_4^2 + d_5^2 + d_6^2) + d_5^2(d_4^2 + 2d_6^2)) - 2d_2 d_3 d_4 d_5 d_6^2 + d_4^2(d_5^2 d_6^2 + d_3^2(d_5^2 + 2d_6^2)))}{d_2^2 d_3^2 d_4^2 d_6^2 d_6^2},$$

$$V_1(x) = \frac{8x^3(d_1 d_6 - d_3 d_4)(d_1 d_2 d_3 d_4 (d_4^2 + d_6^2) - (d_1^2 d_2 d_4^2 - d_3 d_4 (d_1^2 + d_4^2) d_5 + d_2(2d_1^2 + d_4^2) d_6^2) d_6)}{d_1^3 d_2 d_3^2 d_4^4 d_6^2 d_6^2}$$
$$+ \frac{8x(d_3 d_4 d_5 d_6 (d_3 d_4 - d_2 d_5) + d_1(d_2 d_3^2 (d_4^2 + d_6^2) + d_6^2(d_2(d_3^2 + 2d_5^2) - d_3 d_4 d_5)))}{d_1 d_2 d_3^2 d_4^2 d_6^2 d_6^2},$$

$$V_0(x) = 4(d_3^{-2} + d_6^{-2} + d_6^{-2}) + \frac{4x^4(d_4^2 d_5^2 + d_1^2(d_4^2 + d_5^2))(d_3 d_4 - d_1 d_6)^2}{d_1^4 d_3^2 d_4^4 d_6^2 d_6^2}$$
$$- \frac{4x^2(d_4^2(2d_3^2 d_5^2 + d_6^2(d_3^2 + d_5^2)) - 2d_1 d_3 d_4 d_5^2 d_6 + d_1^2(d_6^2(d_4^2 + 2d_5^2) + d_3^2(d_4^2 + d_5^2 + d_6^2)))}{d_1^2 d_3^2 d_4^2 d_6^2 d_6^2}.$$

Appendix B. Computation of $d(1, 2)$ when $1/d_2 = 1/d_5 = \infty$

$$y_1 = \mathrm{sign}(d_1 d_6 - d_3 d_4) d_6 \sqrt{\frac{d_3^2 + d_6^2}{d_4^2 + d_6^2}}, \qquad z_1 = d_3 \frac{(d_1 d_3 + d_4 d_6)}{\sqrt{(d_3 d_4 - d_1 d_6)^2}} \sqrt{\frac{d_4^2 + d_6^2}{d_3^2 + d_6^2}},$$
$$y_2 = \frac{d_1 |d_3 - d_4| \pm |d_4(d_1 + d_6)|}{\sqrt{(d_3 - d_4)^2 + (d_1 + d_6)^2}}, \qquad z_2 = \pm \frac{d_3(d_1 + d_6)}{\sqrt{(d_3 - d_4)^2 + (d_1 + d_6)^2}},$$
$$y_3 = \frac{d_1 |d_3 + d_4| \pm |d_4(d_1 - d_6)|}{\sqrt{(d_3 + d_4)^2 + (d_1 - d_6)^2}}, \qquad z_3 = \pm \frac{d_3(d_1 - d_6)}{\sqrt{(d_3 + d_4)^2 + (d_1 - d_6)^2}}.$$

The choice of the signs depends on the sign of expressions in modules.

Appendix C. Computation of $d(1,3)$ when $1/d_2 = 1/d_5 = \infty$

$$x_0 = \frac{d_1 d_6 \sqrt{d_3^2 + d_6^2}}{d_1 d_6 - d_3 d_4}, \qquad z_0 = \frac{d_3^2}{\sqrt{d_3^2 + d_6^2}}, \qquad x_1 = \frac{d_1^2}{\sqrt{d_1^2 + d_4^2}},$$

$$z_1 = \frac{d_3 d_4 \sqrt{d_1^2 + d_4^2}}{d_3 d_4 - d_1 d_6}, \qquad x_2 = \mathrm{sign}(d_1 d_3 + d_4 d_6) \frac{d_1(d_3 + d_4)}{\sqrt{(d_3 + d_4)^2 + (d_1 - d_6)^2}},$$

$$z_2 = \frac{d_3 \left(d_3 \sqrt{(d_1 d_3 + d_4 d_6)^2} \pm d_6 (d_3 (d_3 + d_4) - d_1 d_6 + d_6^2) \right)}{\sqrt{(d_3 + d_4)^2 + (d_1 - d_6)^2} (d_3^2 + d_6^2)},$$

$$x_3 = \mathrm{sign}(d_1 d_3 + d_4 d_6) \frac{d_1 (d_3 - d_4)}{\sqrt{(d_3 - d_4)^2 + (d_1 + d_6)^2}},$$

$$z_3 = \frac{d_3 \left(d_4 \sqrt{(d_1 d_3 + d_4 d_6)^2} \pm d_6 (d_4 (d_4 - d_3) + d_1 (d_1 + d_6)) \right)}{(d_3 d_4 - d_1 d_6) \sqrt{(d_3 - d_4)^2 + (d_1 + d_6)^2}}.$$

The choice of the signs depends on the sign of expressions in modules.

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